NACA TN 1891

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 1891

(1) NACA-TN-1891

ELASTIC BUCKLING OF A SIMPLY SUPPORTED PLATE UNDER A COMPRESSIVE STRESS THAT VARIES LINEARLY IN THE DIRECTION OF LOADING

By Charles Libove, Saul Ferdman, and John J. Reusch

Langley Aeronautical Laborat
Langley Air Force Base, Va.

Washington
June 1949

34

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 1891

ELASTIC BUCKLING OF A SIMPLY SUPPORTED PLATE UNDER A COMPRESSIVE STRESS THAT VARIES LINEARLY IN THE DIRECTION OF LOADING

By Charles Libove, Saul Ferdman, and John J. Reusch

SUMMARY

Results are presented of calculations for the elastic buckling load of a simply supported flat rectangular plate of uniform thickness subjected to unequal compressive stresses at two opposite edges, with a linear variation of stress between the two edges. The difference between the compressive stresses at the two loaded edges is equilibrated by shear stresses along the other two edges. The results show that a plate with a linear stress gradient will buckle at an average stress that is lower, but at a maximum stress that may be appreciably higher, than the uniform compressive buckling stress of the same plate.

INTRODUCTION

In most studies of the compressive buckling of flat rectangular plates the compressive stress has been taken as uniform throughout the plate. Cases of practical interest exist, however, in which the compression is not uniform but varies from one loaded edge to the other, as, for example, when the plate forms part of the upper or lower skin of an airplane wing in bending.

One case of plate buckling under a load gradient is considered in the present paper. The problem is that of determining the elastic buckling load of a simply supported flat rectangular plate under a compressive stress that varies linearly from one loaded edge to the other, the difference in stress being equilibrated by uniform shears along the other two edges. (See fig. 1.) That this stress condition is a possible one, with regard to both equilibrium and continuity, is shown in the appendix.

Solutions by the Rayleigh-Ritz method are obtained for several length-to-width ratios from 0 to ∞. For each length-to-width ratio several ratios of minimum compressive stress to maximum compressive stress are considered. Some negative values are included among these stress ratios. These values correspond to a minimum stress that is tensile rather than compressive.

The influence of a loading gradient on the compressive buckling of a simply supported plate has been considered previously in reference 1. The loading gradient was exponential and the plate was assumed to have a thickness variation estimated to give minimum weight. Because economy of construction may often preclude the use of taper, the analysis of the constant-thickness plate in the present paper is felt to be of interest.

The results of the present investigation are described in the section entitled "Results and Discussion." The theoretical derivations are included in the appendix.

SYMBOLS

a	plate length measured in compressed direction (x-direction)
b	plate width
β	length-to-width ratio (a/b)
t	plate thickness
E	Young's modulus of elasticity
μ	Poisson's ratio
D	plate flexural stiffness $\left(\frac{Et^3}{12(1 - \mu^2)}\right)$
x, y	coordinates of plate, shown in figure 1
w	lateral deflection
w_{mn}, w_{ij}; w_m, w_i	coefficients of Fourier series for w
σ	longitudinal compressive stress (stress in x-direction)
σ_{max}	longitudinal compressive stress at more highly compressed end when buckling occurs
σ_{min}	longitudinal compressive stress at less highly compressed end when buckling occurs
σ_{av}	longitudinal compressive stress at center of plate when buckling occurs

r load-gradient parameter $\left(\frac{\sigma_{min}}{\sigma_{max}}\right)$

$$\left.\begin{aligned} k &= \frac{\sigma t b^2}{\pi^2 D} \\ k_{max} &= \sigma_{max}\frac{tb^2}{\pi^2 D} \\ k_{av} &= \sigma_{av}\frac{tb^2}{\pi^2 D} \end{aligned}\right\} \text{ dimensionless stress coefficients in terms of } b$$

$$\left.\begin{aligned} k' &= \frac{\sigma t a^2}{\pi^2 D} \\ k'_{max} &= \sigma_{max}\frac{ta^2}{\pi^2 D} \\ k'_{av} &= \sigma_{av}\frac{ta^2}{\pi^2 D} \end{aligned}\right\} \text{ dimensionless stress coefficients in terms of } a$$

F potential energy

σ_x, σ_y middle-plane direct stresses in x-direction and y-direction, respectively, prior to buckling, positive for tension

τ_{xy} middle-plane shear stress in x-direction and y-direction prior to buckling

$$\left.\begin{matrix} i, j, \\ m, n, \\ p, q, r \end{matrix}\right\} \text{ integers; also used as subscripts}$$

Primes (') and double primes (") are used with subscripts i, m, n, p, and r to indicate odd and even values of the subscripts, respectively.

RESULTS AND DISCUSSION

Graphical Presentation

The results of the investigation are given graphically in figures 2 to 5. In the first three of these figures the solid circles represent calculated points. The curves were obtained by fairing through the calculated points. There is reason to believe that these curves should theoretically contain slight discontinuities in slope corresponding to sudden changes in the buckle pattern. The fairing of smooth curves, however, was felt to be justified by practical considerations.

The maximum compressive stress in the plate when buckling occurs is given in figure 2 for several different values of length-width ratio β where

$$\beta = \frac{a}{b} \tag{1}$$

and a is the plate length in the compressed direction and b, plate width. The stress appears in the parameter k_{max}, defined as

$$k_{max} = \sigma_{max} \frac{tb^2}{\pi^2 D} \tag{2}$$

where

σ_{max} compressive stress at the more highly compressed end when buckling occurs

t plate thickness

D plate flexural stiffness $\left(D = \dfrac{Et^3}{12(1 - \mu^2)}\right)$

E Young's modulus

μ Poisson's ratio

For each value of β, k_{max} is plotted as a function of r, the ratio of the minimum compressive stress σ_{min} to the maximum compressive stress σ_{max}. This ratio is usually a known constant independent of the absolute magnitude of the load. Negative values of r correspond to tensile values of σ_{min}.

Figure 2 cannot contain a curve for $\beta = 0$ ($b = \infty$) because the parameter k_{max} involves b. The curves are therefore replotted in figure 3 in terms of k'_{max}, where

$$k'_{max} = \sigma_{max} \frac{ta^2}{\pi^2 D} \tag{3}$$

Figures 2 and 3 show that the presence of a stress gradient ($r \neq 1$) permits the plate to develop a σ_{max} that is greater than the uniform stress required for buckling ($r = 1$). For a given value of r this effect becomes less pronounced as β increases. The effect vanishes completely for $\beta = \infty$.

Although the presence of a stress gradient permits the maximum stress in the plate to exceed its critical uniform value, it causes a reduction in the average stress at which buckling occurs. This reduction in the average stress is illustrated in figure 4 through use of the parameter k_{av}, defined as

$$k_{av} = \sigma_{av} \frac{tb^2}{\pi^2 D} \tag{4}$$

where σ_{av} is the longitudinal compressive stress at the center of the plate when buckling occurs.

The complete longitudinal direct-stress picture at buckling is shown in figure 5 for several values of β. The longitudinal compressive stress σ is plotted in terms of k for β other than zero and k' for $\beta = 0$, where

$$k = \frac{\sigma t b^2}{\pi^2 D} \tag{5}$$

$$k' = \frac{\sigma t a^2}{\pi^2 D} \tag{6}$$

Figure 5 shows at the same time the reduction in the average critical stress and the increase in the maximum critical stress produced by a stress gradient, and how the increase in the maximum critical stress is diminished as the length-to-width ratio increases.

Tabular Presentation

The basic calculated information from which the curves of figures 2 to 5 were obtained is given in table 1. In the first part of the table, calculated values of k_{av} are tabulated against given values of r and β for the general case ($\beta \neq 0$). The values of k_{max} and k'_{max} plotted in figures 2 and 3 were obtained from this basic information through the relationships

$$k_{max} = \frac{2}{1 + r} k_{av} \tag{7}$$

$$k'_{max} = k_{max}\beta^2 \tag{8}$$

In the second part of the table calculated values of k'_{av} for the infinitely wide plate ($\beta = 0$) are tabulated against r, where k'_{av} is defined by the equation

$$k'_{av} = \sigma_{av} \frac{ta^2}{\pi^2 D} \tag{9}$$

The values of k'_{max} plotted in figure 3 for the infinitely wide plate were obtained from the tabulated values of k'_{av} through the relationship

$$k'_{max} = \frac{2}{1 + r} k'_{av} \tag{10}$$

CONCLUSIONS

A theoretical investigation has been made of the buckling of a simply supported flat rectangular plate of uniform thickness under a compressive stress that varies linearly in the direction of loading. The parameters in terms of which the buckling stresses are plotted are the length-to-width ratio β and the ratio of minimum compressive stress to maximum compressive stress r.

The main conclusion to be drawn from the investigation is that when buckling occurs the stress at the more highly loaded end of the plate may be appreciably greater than the uniform buckling stress of the same plate. For a given value of r, this effect becomes more pronounced as β decreases. At the same time the average stress in the plate at buckling will be less than the uniform buckling stress.

Langley Aeronautical Laboratory
National Advisory Committee for Aeronautics
Langley Air Force Base, Va., April 4, 1949

APPENDIX

THEORETICAL DERIVATION

General Case: $\beta \neq 0$ or ∞

Energy expression.– Of all conceivable buckle patterns satisfying boundary conditions the actual buckle pattern is that for which the potential energy F, as given by the following expression, is a minimum (reference 2):

$$F = \frac{D}{2}\int_0^b \int_0^a \left\{\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\mu)\left[\frac{\partial^2 w}{\partial x^2}\,\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]\right\} dx\,dy$$

$$+ \frac{t}{2}\int_0^b \int_0^a \left[\sigma_x\left(\frac{\partial w}{\partial x}\right)^2 + \sigma_y\left(\frac{\partial w}{\partial y}\right)^2 + 2\tau_{xy}\,\frac{\partial w}{\partial x}\,\frac{\partial w}{\partial y}\right] dx\,dy \qquad \text{(A1)}$$

where

x, y coordinates of the plate, shown in figure 1

w lateral deflection

σ_x, σ_y middle–plane direct stresses in x–direction and y–direction, respectively, prior to buckling, positive for tension

τ_{xy} middle–plane shear stress in x–direction and y–direction prior to buckling

Middle-plane stresses.- Algebraic expressions for the middle-plane stresses σ_x, σ_y, and τ_{xy}, needed in equation (A1), can be obtained by solving the plane-stress-equilibrium equations

$$\left.\begin{aligned}\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} = 0\\[2ex] \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{xy}}{\partial x} = 0\end{aligned}\right\} \qquad \text{(A2)}$$

First, the following expression for σ_x, corresponding to a linear variation of compressive stress between σ_{min} and σ_{max}, is written

$$\sigma_x = -\sigma_{av}\left[\left(\frac{1 - r}{1 + r}\right)\frac{2x}{a} + \frac{2r}{1 + r}\right] \qquad \text{(A3)}$$

where

σ_{av} compressive stress in the x-direction at plate center $x = \frac{a}{2}$

$r = \frac{\sigma_{min}}{\sigma_{max}} = \frac{\text{Minimum compressive stress}}{\text{Maximum compressive stress}}$

a plate length

Substitution of the expression for σ_x from equation (A3) in equations (A2) permits these equations to be integrated for τ_{xy} and σ_y. This integration, subject to the conditions that no transverse compressive stress and no component of pure shear exist on the plate, gives the following expressions for τ_{xy} and σ_y:

$$\tau_{xy} = \frac{\sigma_{av}}{\beta}\left(\frac{1 - r}{1 + r}\right)\left(\frac{2y}{b} - 1\right) \qquad \text{(A4)}$$

$$\sigma_y = 0$$

Through equations (A3) to (A5) the middle-plane stresses have all been expressed in terms of a single stress parameter σ_{av} and the load-gradient parameter r.

Expressions (A3) to (A5), beside satisfying equilibrium conditions, also satisfy the condition of compatibility (reference 3) and, therefore, represent a physically possible set of stresses for an elastic sheet.

Deflection function.- The buckle pattern of the plate may be represented by the double Fourier series

$$w = \sum_{m=1}^{M} \sum_{n=1}^{N} w_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{A6}$$

which satisfies the boundary conditions for simple support term by term. Any arbitrary degree of accuracy is in principle possible by letting M and N approach infinity. For practical purposes, however, the use of a finite number of selected terms in equation (A6) is sufficient. Terms that lie within the range $m = 1$ to M and $n = 1$ to N but that are not used may be thought of as having been assigned zero values for the coefficient w_{mn}.

The coefficients w_{mn} are at present undetermined. Their relative magnitudes are subsequently chosen subject to the condition that F, as given by expression (A1), be a minimum.

Stability determinant.- Substituting equations (A3) to (A6) in expression (A1) and performing the indicated integrations give

$$F = \frac{\pi^2 D}{ab} k_{av} \left\{ \sum_{m=1}^{M} \sum_{n=1}^{N} w_{mn}^2 \frac{\pi^2}{8} \left[\frac{1}{k_{av}} \left(\frac{m^2}{\beta} + \beta n^2 \right)^2 - m^2 \right] \right.$$

$$\left. + 2\left(\frac{1 - r}{1 + r}\right) \sum_{m=1}^{M} \sum_{n=1}^{N} \sum_{\substack{p=1 \\ m \pm p = \text{odd}}}^{M} w_{mn} w_{pn} \frac{m^3 p}{\left(m^2 - p^2\right)^2} \right.$$

$$\left. - 4\left(\frac{1 - r}{1 + r}\right) \sum_{\substack{m=1 \\ m \pm p = \text{odd} \\ n \pm q = \text{even} \\ q \neq n}}^{M} \sum_{n=1}^{N} \sum_{p=1}^{M} \sum_{q=1}^{N} w_{mn} w_{pq} \frac{mnpq}{\left(q^2 - n^2\right)\left(m^2 - p^2\right)} \right\} \tag{A7}$$

where

$$k_{av} = \sigma_{av} \frac{tb^2}{\pi^2 D} \tag{A8}$$

The deflection coefficients w_{mn} must now be chosen such that F is a minimum; that is, they must satisfy the equations

$$\frac{\partial F}{\partial w_{ij}} = 0 \qquad \begin{pmatrix} i = 1, 2, 3, \ldots M \\ j = 1, 2, 3, \ldots N \end{pmatrix} \tag{A9}$$

In expanded form, equations (A9) can be written

$$w_{ij} \frac{\pi^2}{8} \left[\frac{1}{k_{av}}\left(\frac{i^2}{\beta} + \beta j^2\right)^2 - i^2\right] + \left(\frac{1 - r}{1 + r}\right) \sum_{\substack{p=1 \\ p \pm i = \text{odd}}}^{M} w_{pj} \frac{ip\left(i^2 + p^2\right)}{\left(i^2 - p^2\right)^2}$$

$$- 4 \left(\frac{1 - r}{1 + r}\right) \sum_{\substack{p=1 \\ p \pm i = \text{odd} \\ q \pm j = \text{even} \\ q \neq j}}^{M} \sum_{q=1}^{N} w_{pq} \frac{ijpq}{\left(q^2 - j^2\right)\left(i^2 - p^2\right)} = 0 \tag{A10}$$

$$\begin{pmatrix} i = 1, 2, 3, \ldots M \\ j = 1, 2, 3, \ldots N \end{pmatrix}$$

Equations (A10) constitute a set of simultaneous equations to be solved for the w's. Since these equations are homogeneous, they can have nonzero solutions for the w's – that is, the plate can buckle – only if the determinant formed by the coefficients of the w's equals zero. This determinant contains k_{av}. The lowest value of k_{av} for which the determinant vanishes determines the buckling load. The order of the determinant is equal to the number of terms used in the deflection function (equation (A6)).

The accuracy of a solution thus obtained may be judged in two ways. The first consists in repeating the calculations several times, each time including additional terms in the deflection function (A6), and noting the sizes of successive changes in k_{av}. When these changes become negligible, the solution may be assumed to be converged. The correct buckle pattern, that is, the one for which F is a minimum, can also be shown to be the buckle pattern for which k_{av} is a minimum. Of several buckle patterns, therefore, the most accurate is the one giving the lowest k_{av}. The successive changes in k_{av} should therefore be negative. The second way of judging the accuracy consists in calculating the relative values of the coefficients w_{mn} (their absolute values are indeterminate). The maximum w_{mn} will occur for a particular set of values of m and n, say m_o and n_o, and the magnitudes of the other w_{mn}'s will generally decrease as m and n become farther and farther from m_o and n_o. The omission of any important terms from the deflection function (A6) can be noted by systematically tabulating the relative values of the coefficients.

Simplifications to stability determinant.- The preceding paragraph describes in principle the use of the stability determinant to calculate buckling stresses. Two simplifications to the determinant are feasible and should be considered before calculations are made. The first simplification follows from the fact that the equations represented by equations (A10) fall into two completely independent subsets, one corresponding to even values of j and the other, to odd values of j. The first subset contains only those w's for which the second subscript is even and the other subset contains only those w's for which the second subscript is odd. The significance of the subscripts in the deflection function (equation (A6)) indicates that the first subset corresponds to buckling that is antisymmetrical about the line $y = \frac{b}{2}$ and the second, to buckling that is symmetrical about that line. Calculations show that the second subset gives the lower buckling loads, and attention is hereinafter confined to it. This subset is given as follows:

$$w_{ij}\,\frac{\pi^2}{8}\left[\frac{1}{k_{av}}\left(\frac{i^2}{\beta}+\beta j^2\right)^2 - i^2\right] + \left(\frac{1-r}{1+r}\right)\sum_{\substack{p=1\\ p\pm i=\text{odd}}}^{M} w_{pj}\,\frac{ip\left(i^2+p^2\right)}{\left(i^2-p^2\right)^2}$$

$$-4\left(\frac{1-r}{1+r}\right)\sum_{\substack{p=1\\ p\pm i=\text{odd}}}^{M}\ \sum_{\substack{q=1,3,5\\ q\neq j}}^{N} w_{pq}\,\frac{ijpq}{\left(q^2-j^2\right)\left(i^2-p^2\right)} = 0 \qquad \text{(A11)}$$

$$\begin{pmatrix} i = 1, 2, 3, \ldots M \\ j = 1, 3, 5, \ldots N \end{pmatrix}$$

These equations, instead of equations (A10), can now be used to write a stability determinant.

A second simplification is possible by noting that equations (A11) can be further divided into two subsets, one corresponding to even values of i and the other, to odd values of i. These two sets are written as follows, with primes on the odd subscripts and double primes on the even ones to make them more readily distinguishable. (Since j and q are always odd, the distinguishing mark is omitted from these subscripts.)

$$w_{i'j}A_{i'j} + \sum_{p''=2,4,6\ldots}^{M} w_{p''j}B_{i'p''}$$

$$-\sum_{p''=2,4,6\ldots}^{M}\ \sum_{\substack{q=1,3,5\ldots\\ q\neq j}}^{N} w_{p''q}C_{i'jp''q} = 0 \qquad \text{(A12a)}$$

$$\begin{pmatrix} i' = 1, 3, 5, \ldots M \\ j = 1, 3, 5, \ldots N \end{pmatrix}$$

and

$$w_{i''j}A_{i''j} + \sum_{p'=1,3,5\ldots}^{M} w_{p'j}B_{i''p'}$$

$$- \sum_{p'=1,3,5\ldots}^{M} \sum_{\substack{q=1,3,5\ldots \\ q\neq j}}^{N} w_{p'q}C_{i''jp'q} = 0 \qquad \text{(A12b)}$$

$$\begin{pmatrix} i'' = 2,\ 4,\ 6,\ \ldots\ M \\ j = 1,\ 3,\ 5,\ \ldots\ N \end{pmatrix}$$

where

$$A_{ij} = \frac{\pi^2}{8}\left[\frac{1}{k_{av}}\left(\frac{i^2}{\beta} + \beta j^2\right) - i^2\right]$$

$$B_{ip} = \left(\frac{1 - r}{1 + r}\right)\frac{ip\left(i^2 + p^2\right)}{\left(i^2 - p^2\right)^2}$$

$$C_{ijpq} = 4\left(\frac{1 - r}{1 + r}\right)\frac{ijpq}{\left(q^2 - j^2\right)\left(i^2 - p^2\right)}$$

Equations (A12a) and equations (A12b) are not independent since the same w's appear in both sets; however, they afford the possibility of eliminating roughly half of the unknowns. For example, equations (A12b) may be solved for each coefficient of the type $w_{m''n}$ in terms of all the coefficients of the type $w_{m'n}$ and the resulting expressions used to eliminate all the coefficients of the type $w_{m''n}$ in equations (A12a). Equations (A12a) then become a set of equations involving only the coefficients of the type $w_{m'n}$ explicitly. The buckling stress is obtained as before by finding the lowest k_{av} for which the stability determinant of these equations vanishes. If desired, equations (A12a) and (A12b) can be used to furnish an alternate set of equations in which only the coefficients of the type $w_{m''n}$ appear explicitly.

The just described reduction of equations (A12a) and (A12b) to a single set of equations involving only the coefficients of the type $w_{m'n}$ is now shown in detail. Equations (A12b) are first solved for $w_{i''j}$ to obtain

$$w_{i''j} = -\sum_{p'=1,3,5,\ldots}^{M} w_{p'j}\frac{B_{i''p'}}{A_{i''j}} + \sum_{p'=1,3,5,\ldots}^{M}\ \sum_{\substack{q=1,3,5,\ldots\\ q\neq j}}^{N} w_{p'q}\frac{C_{i''jp'q}}{A_{i''j}} \tag{A13}$$

$$\begin{pmatrix} i'' = 2, 4, 6, \ldots M \\ j = 1, 3, 5, \ldots N \end{pmatrix}$$

In order to avoid confusion when equations (A13) are substituted in equations (A12a), the index q in equations (A13) is changed to r. Furthermore, in order to obtain a general formula for the coefficient of the type $w_{m''n}$, the indices i'' and j are changed to m'' and n, respectively. The result is

$$w_{m''n} = -\sum_{p'=1,3,5,\ldots}^{M} w_{p'n}\frac{B_{m''p'}}{A_{m''n'}} + \sum_{p'=1,3,5,\ldots}^{M}\ \sum_{\substack{r=1,3,5,\ldots\\ r\neq n}}^{N} w_{p'r}\frac{C_{m''np'r}}{A_{m''n}} \tag{A14}$$

$$\begin{pmatrix} m'' = 2, 4, 6, \ldots M \\ n = 1, 3, 5, \ldots N \end{pmatrix}$$

In equations (A12a) the coefficients $w_{p''j}$ and $w_{p''q}$ are of the type $w_{m''n}$ and may therefore be eliminated by the use of equations (A14). Equations (A12a) then become

$$w_{i'j}A_{i'j} + \sum_{p''=2,4,6,\ldots}^{M} B_{i'p''}\left(-\sum_{p'=1,3,5,\ldots}^{M} w_{p'j}\frac{B_{p''p'}}{A_{p''j}}\right.$$

$$\left.+\sum_{p'=1,3,5,\ldots}^{M}\ \sum_{\substack{r=1,3,5,\ldots\\ r\neq j}}^{N} w_{p'r}\frac{C_{p''jp'r}}{A_{p''j}}\right)$$

$$-\sum_{p''=2,4,6,\ldots}^{M}\ \sum_{\substack{q=1,3,5,\ldots\\ q\neq j}}^{N} C_{i'jp''q}\left(-\sum_{p'=1,3,5,\ldots}^{M} w_{p'q}\frac{B_{p''p'}}{A_{p''q}}\right.$$

$$\left.+\sum_{p'=1,3,5,\ldots}^{M}\ \sum_{\substack{r=1,3,5,\ldots\\ r\neq q}}^{N} w_{p'r}\frac{C_{p''qp'r}}{A_{p''q}}\right) = 0$$

$$\begin{pmatrix} i' = 1, 3, 5, \ldots M \\ j = 1, 3, 5, \ldots N \end{pmatrix}$$

or, by removing brackets,

$$
\begin{aligned}
& w_{i'j}A_{i'j} - \sum_{p''=2,4,6,\ldots}^{M} \; \sum_{p'=1,3,5,\ldots}^{M} w_{p'j} \frac{B_{i'p''}B_{p''p'}}{A_{p''j}} \\
& + \sum_{p''=2,4,6,\ldots}^{M} \; \sum_{p'=1,3,5,\ldots}^{M} \; \sum_{\substack{r=1,3,5,\ldots \\ r\neq j}}^{N} w_{p'r} \frac{B_{i'p''}C_{p''jp'r}}{A_{p''j}} \\
& + \sum_{p''=2,4,6,\ldots}^{M} \; \sum_{\substack{q=1,3,5,\ldots \\ q\neq j}}^{N} \; \sum_{p'=1,3,5,\ldots}^{M} w_{p'q} \frac{B_{p''p'}C_{i'jp''q}}{A_{p''q}} \\
& - \sum_{p''=2,4,6,\ldots}^{M} \; \sum_{\substack{q=1,3,5,\ldots \\ q\neq j}}^{N} \; \sum_{p'=1,3,5,\ldots}^{M} \; \sum_{\substack{r=1,3,5,\ldots \\ r\neq q}}^{N} w_{p'r} \frac{C_{i'jp''q}C_{p''qp'r}}{A_{p''q}} = 0 \qquad \text{(A15)}
\end{aligned}
$$

$$
\begin{pmatrix} i' = 1,\ 3,\ 5,\ \ldots\ M \\ j = 1,\ 3,\ 5,\ \ldots\ N \end{pmatrix}
$$

A stability determinant is furnished by the coefficients of the w's in equations (A15).

The main advantage of equations (A15) over equations (A11) lies in the fact that, no matter how many w's are included in the deflection function (A6), the order of the stability determinant will be only equal to the number of w's of the $w_{m'n}$ type (first subscript odd, second odd); whereas, if equations (A11) are used they furnish a stability determinant the order of which is equal to the total number of w's. The main disadvantages of equations (A15) are that each term in the stability determinant is more complicated and that, if the relative magnitudes of the w's are desired, a separate calculation for $w_{m''n}$ coefficients (by using equations (A14)) is required after the $w_{m'n}$ coefficients have been determined.

Calculations.- Equations (A11) rather than equations (A15) were chosen as the basis of a stability determinant. The matrix iteration method (reference 4) was used to obtain the maximum value of $1/k_{av}$ (corresponding to the minimum value of k_{av} and therefore the lowest buckling load) for which nonzero solutions for the w's are possible. The matrix iteration method at the same time furnished the relative magnitudes of the w's and thus permitted the judgment of whether or not all the important deflection coefficients had been included in the deflection function (A6). The results of the calculations are summarized in table 2 which gives, for different assumed values of r and β, the calculated values of k_{av} and the relative magnitudes of the w_{mn}'s.

Special Case: β = 0

For the special case of an infinitely wide plate ($b = \infty$ and $\beta = 0$), equation (A6) has no meaning and the solution obtained is not valid. For this case, the plate will be assumed to buckle as a column, with straight lines in the y-direction. This buckle pattern can be represented by the single Fourier series

$$w = \sum_{m=1}^{M} w_m \sin \frac{m\pi x}{a} \tag{A16}$$

Equation (A16) is substituted in the energy expression (A1) in place of equation (A6). The y integration in equation (A1) is now performed over

a unit width instead of from 0 to b. Minimization of the potential energy as before gives the following system of equations to replace equations (A11):

$$w_i \frac{\pi^2}{8} i^2\left(\frac{i^2}{k'_{av}} - 1\right) - \left(\frac{1 - r}{1 + r}\right) \sum_{\substack{p=1 \\ p\pm i=\text{odd}}}^{M} w_p \frac{ip\left(i^2 + p^2\right)}{\left(i^2 - p^2\right)^2} = 0 \qquad \text{(A17)}$$

$$\left(i = 1, 2, 3, \ldots M\right)$$

The above system of equations was solved by matrix iteration to obtain k'_{av} and the relative magnitudes of the w_m's used in equation (A16). The results are given in table 3.

Special Case: $\beta = \infty$

As for the case of $\beta = 0$, the buckle pattern of an infinitely long plate ($a = \beta = \infty$) cannot be represented by equation (A6). The solution for this case is, however, readily obtainable by physical reasoning.

Consider a plate having a length a much greater than its width b, with a longitudinal compressive stress at buckling varying from σ_{min} at one end to σ_{max} at the other end. (See fig. 6.) The calculations for the general case ($\beta \neq 0$) have shown that σ_{max} will be greater than, and σ_{min} less than, the uniform buckling stress of the plate, which in this case is $\frac{4\pi^2 D}{b^2 t}$. (See dash line in fig. 6.) The length of plate in which the longitudinal compressive stress exceeds the uniform buckling stress is denoted by d in figure 6.

Now assume that $r = \frac{\sigma_{min}}{\sigma_{max}}$ remains constant while a and, therefore, β approach infinity. If σ_{max} remained constant, the length d would soon be many times the buckle wave length of the uniformly compressed plate. It is not physically plausible that a plate can sustain a stress

greater than its uniform buckling stress over so great a length. Hence, for a given value of r, the following conclusion can be drawn:

$$\sigma_{max} \rightarrow \frac{4\pi^2 D}{b^2 t}$$

or

$$k_{max} \rightarrow 4 \tag{A18}$$

as $\beta \rightarrow \infty$.

Also, since

$$k_{av} = \frac{rk_{max} + k_{max}}{2}$$

$$k_{av} \rightarrow 2(r + 1) \tag{A19}$$

as $\beta \rightarrow \infty$.

REFERENCES

1. Pines, Samuel, and Gerard, George: Instability Analysis and Design of an Efficiently Tapered Plate under Compressive Loading. Jour. Aero. Sci., vol. 14, no. 10, Oct. 1947, pp. 594–599.

2. Timoshenko, S.: Theory of Elastic Stability. McGraw-Hill Book Co., Inc. 1936, pp. 325–327.

3. Timoshenko, S.: Theory of Elasticity. First ed., McGraw-Hill Book Co., Inc., 1934, equation (22).

4. Von Kármán, Theodore, and Biot, Maurice A.: Mathematical Methods in Engineering. First ed., McGraw-Hill Book Co., Inc., 1940, pp. 196–204.

TABLE 1

CALCULATED RESULTS FOR k_{av} AND k'_{av}

β \ r	−1/3	−1/5	0	1/5	1/2	1
			k_{av}			
0.5	4.55	5.04	5.58	5.91	6.16	6.25
.6	3.66	----	4.53	----	5.05	5.14
.7	3.13	----	3.94	----	4.44	4.53
1	2.43	2.77	3.20	3.53	3.85	4
$\sqrt{2}$	2.11	----	2.92	----	3.86	4.49
2	1.93	2.25	2.71	3.12	3.63	4
3	1.76	2.07	2.51	2.92	3.47	4
5	1.63	1.93	2.36	2.77	3.33	4
∞	1.33	1.60	2	2.40	3	4
			k'_{av}			
0	0.823	0.884	0.941	0.972	0.993	1

TABLE 2

CALCULATED VALUES OF w_{mn} AND CORRESPONDING VALUES OF k_{av}

[General case: $\beta \neq 0$ or ∞]

β	r	n	w_{mn}				k_{av}
			m = 1	m = 2	m = 3	m = 4	
0.5	–1/3	1 3	1 –0.020	–0.160 –.0524	0.0103 .00705	–0.00266	4.55
	–1/5	1 3	1 –0.0145	–0.136 –.0445	0.00741 .0050	–0.0021	5.04
	0	1 3	1 –0.0082	–0.103 –.0335	0.00418 .00284	–0.00151	5.58
	1/5	1 3	1 –0.0041	–0.074 –.024	0.0021		5.91
	1/2	1 3	1 –0.0011	–0.037 –.0126	0.0006 .00040	–0.00053	6.16
0.6	–1/3	1 3	1 –0.01748	–0.18427 –.044203	0.014519 .0071321	–0.0032862	3.66
	0	1 3	1 –0.0073	–0.121 –.0285	0.0062 .00297	–0.00184	4.53
	1/2	1 3	1 –0.0011	–0.047 –.011	0.0009		5.05
0.7	–1/3	1 3	1 –0.0156	–0.214 –.0373	0.0204 .0074	–0.0042	3.13
	0	1 3	1 –0.0068	–0.145 –.0245	0.0091 .00319	–0.00228	3.94
	1/2	1 3	1 –0.0010	–0.058 –.0094	0.0014		4.44
1	–1/3	1 3 5	1 –0.0125	–0.336 –.0226 –.00207	0.054 .00835	–0.0096	2.43
	–1/5	1 3 5	1 –0.0098	–0.31 –.020 –.0019	0.044 .0066	–0.008	2.77

NACA

TABLE 2 – Continued

CALCULATED VALUES OF w_{mn} AND CORRESPONDING VALUES OF k_{av} – Continued

β	r	n	w_{mn} m = 1	m = 2	m = 3	m = 4	m = 5	m = 6	k_{av}
1	0		First Approximation						
		1 3	1 –0.00654	–0.259 –.0156	0.0297	–0.00532			3.21
			Second Approximation						
		1 3 5	1 –0.00644	–0.259 –.0158 –.00154	0.0296 .00437	–0.00518			3.20
	1/5		First Approximation						
		1 3	1	–0.206 –.0120	0.0174				3.54
			Second Approximation						
		1 3	1 –0.00390	–0.209 –.0120	0.0182 .00265	–0.00336			3.53
	1/2		First Approximation						
		1 3	1	–0.123 –.00672	0.0061 .00086	–0.0015			3.85
			Second Approximation						
		1 3 5	1 –0.0013 –.00013	–0.12 –.0067	0.006				3.85
$\sqrt{2}$	–1/3	1 3	1 –0.0117	–0.607 –.0102	0.176 .0104	–0.039 –.0055	0.0098	–0.0027	2.11
	0	1 3	1 –0.00779	–0.584 –.00788	0.14 .00724	–0.024 –.00343	0.0058	–0.0016	2.92

NACA

TABLE 2 – Continued

CALCULATED VALUES OF w_{mn} AND CORRESPONDING VALUES OF k_{av} – Continued

β	r	n	w_{mn}							k_{av}
			m = 1	m = 2	m = 3	m = 4	m = 5	m = 6	m = 7	
$\sqrt{2}$	1/2	1 3	1 −0.00324	−0.550 −.0040	0.0715 .00313	−0.00785 −.00120	0.00201	−0.000518		3.86
2	−1/3	1 3	−0.845 .0102	1 −0.00103	−0.529 −.0086	0.182 .00430	−0.0547	0.0147		1.93
	−1/5	1 3	−0.798 .00887	1 −0.0010	−0.513 −.00791	0.164 .0066	−0.05	0.012		2.25
	0	1 3	−0.718 .0070	1 −0.0009	−0.48 −.0067	0.135 .0050	−0.03	0.0085		2.71
	1/5	1 3	−0.62 .005	1	−0.44 −.005	0.11 .0036	−0.024			3.12
	1/2	1 3	−0.42 .0030	1	−0.34 −.0035	0.05 .0016	−0.01			3.63
3	−1/3	1 3	0.380 −.00469	−0.926 .00596	1	−0.656 −.0048	0.31 .00571	−0.118	0.039	1.76
	−1/5	1 3	0.34 −.004	−0.90 .0053	1	−0.64 −.0044	0.29 .0051	−0.10	0.033	2.07
	0	1 3	0.27 −.0030	−0.85 .0044	1	−0.62 −.0037	0.26 .0041	−0.084	0.025	2.51
	1/5	1 3	0.20 −.0022	−0.78 .0035	1 −0.00028	−0.59 −.0031	0.21 .0031	−0.058		2.92
	1/2	1 3	0.11 −.0011	−0.64 .0021	1 −0.0002	−0.50 −.0021	0.13	−0.029		3.47

NACA

TABLE 2 – Concluded

CALCULATED VALUES OF w_{mn} AND CORRESPONDING VALUES OF k_{av} – Concluded

β	r	n	w_{mn}										k_{av}
			m = 1	m = 2	m = 3	m = 4	m = 5	m = 6	m = 7	m = 8	m = 9	m = 10	
5	–1/3	1	0.084	–0.31	0.66	–0.97	1	–0.77	0.48	–0.26	0.11	–0.05	1.63
	–1/5	1	0.07	–0.27	0.62	–0.95	1	–0.76	0.46	–0.23	0.1	–0.04	1.93
	0	1	0.053	–0.22	0.55	–0.92	1	–0.75	0.43	–0.20	0.072		2.36
		3			–.0025	.0021		–.0018	.0025				
	1/5	1	0.037	–0.16	0.48	–0.89	1	–0.73	0.38	–0.16	0.05		2.77
		3			–.0019	.0017		–.0015	.0020				
	1/2	1	0.018	–0.083	0.33	–0.80	1	–0.67	0.28	–0.09			3.33
		3	–.00026	.00061	.0011	.0011	–.00005	–.0011	.0012	–.001			

NACA

TABLE 3

CALCULATED VALUES OF w_m AND CORRESPONDING VALUES OF k'_{av}

[Special case: $\beta = 0$]

r	w_1	w_2	w_3	w_4	k'_{av}
-1/3	1	0.12	0.0069	0.0019	0.823
-1/5	1	.097	.0046	.0015	.884
0	1	.070	.0023	.0010	.941
1/5	1	.048	.0011	.00068	.972
1/2	1	.025	.00029	.00034	.993

NACA

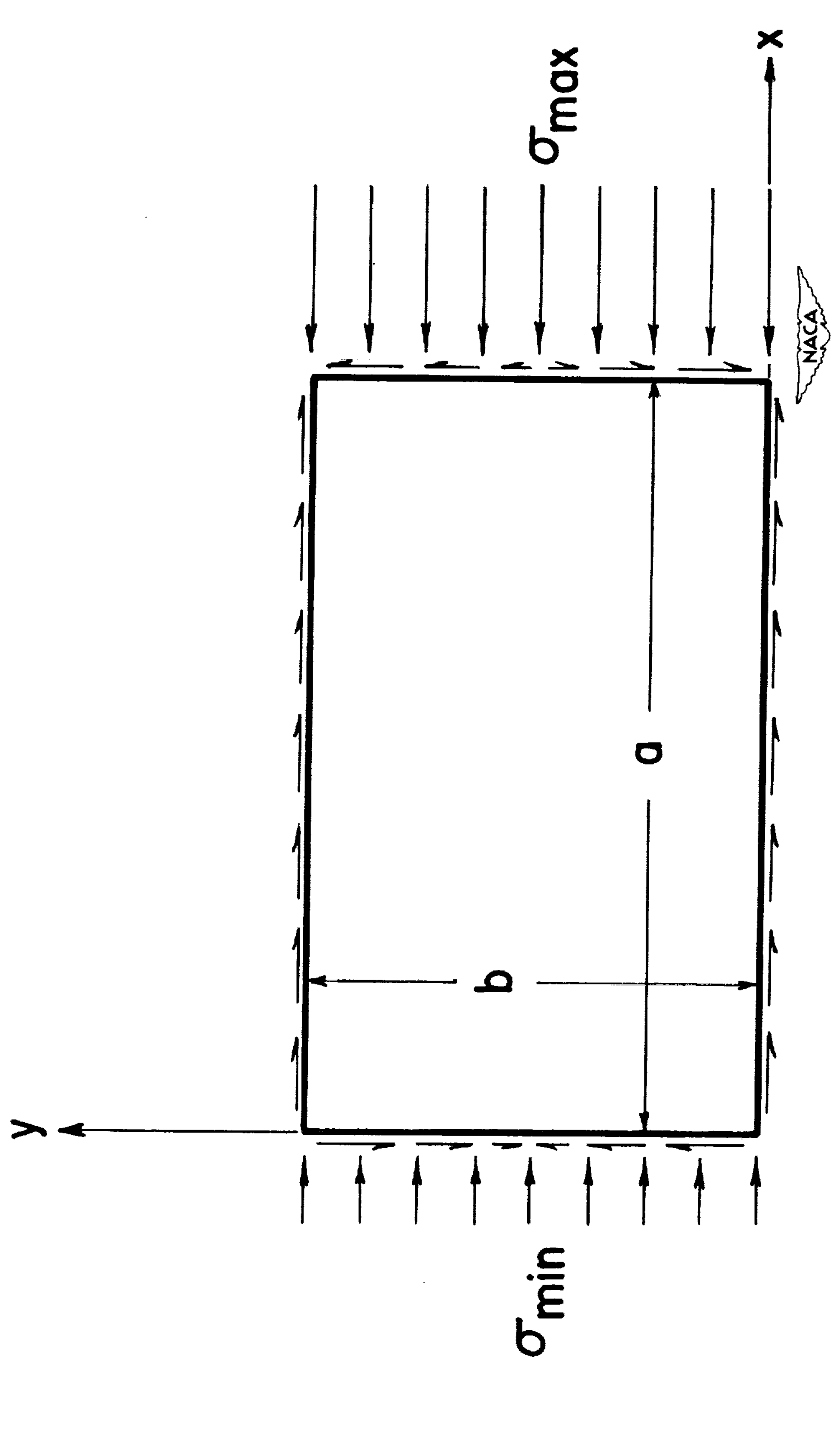

Figure 1.- Plate subjected to unequal compressive stresses along two opposite edges.

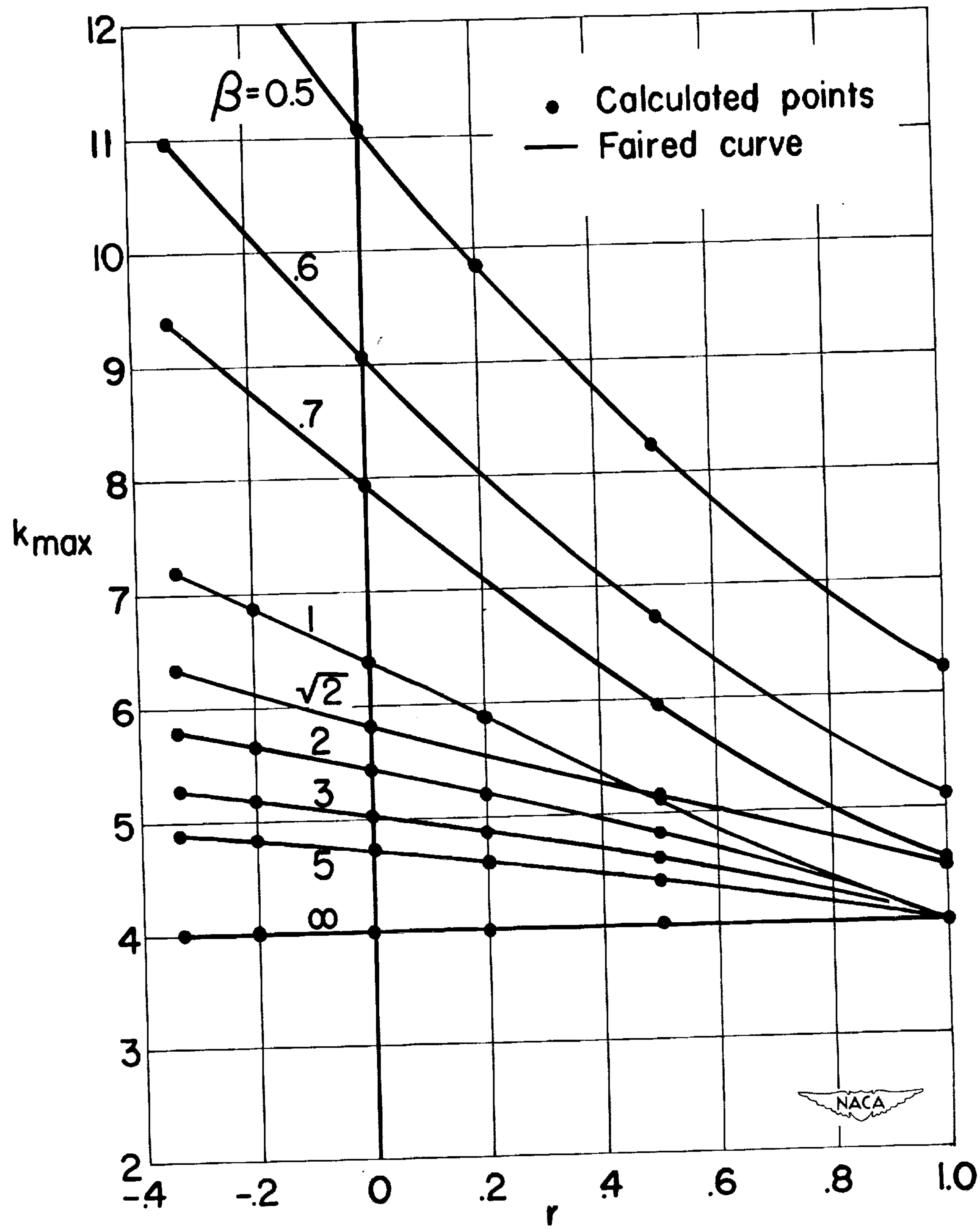

Figure 2.– Coefficient k_{max} used in calculating buckling stress σ_{max}. $\sigma_{max} = k_{max}\left(\frac{\pi^2 D}{b^2 t}\right)$

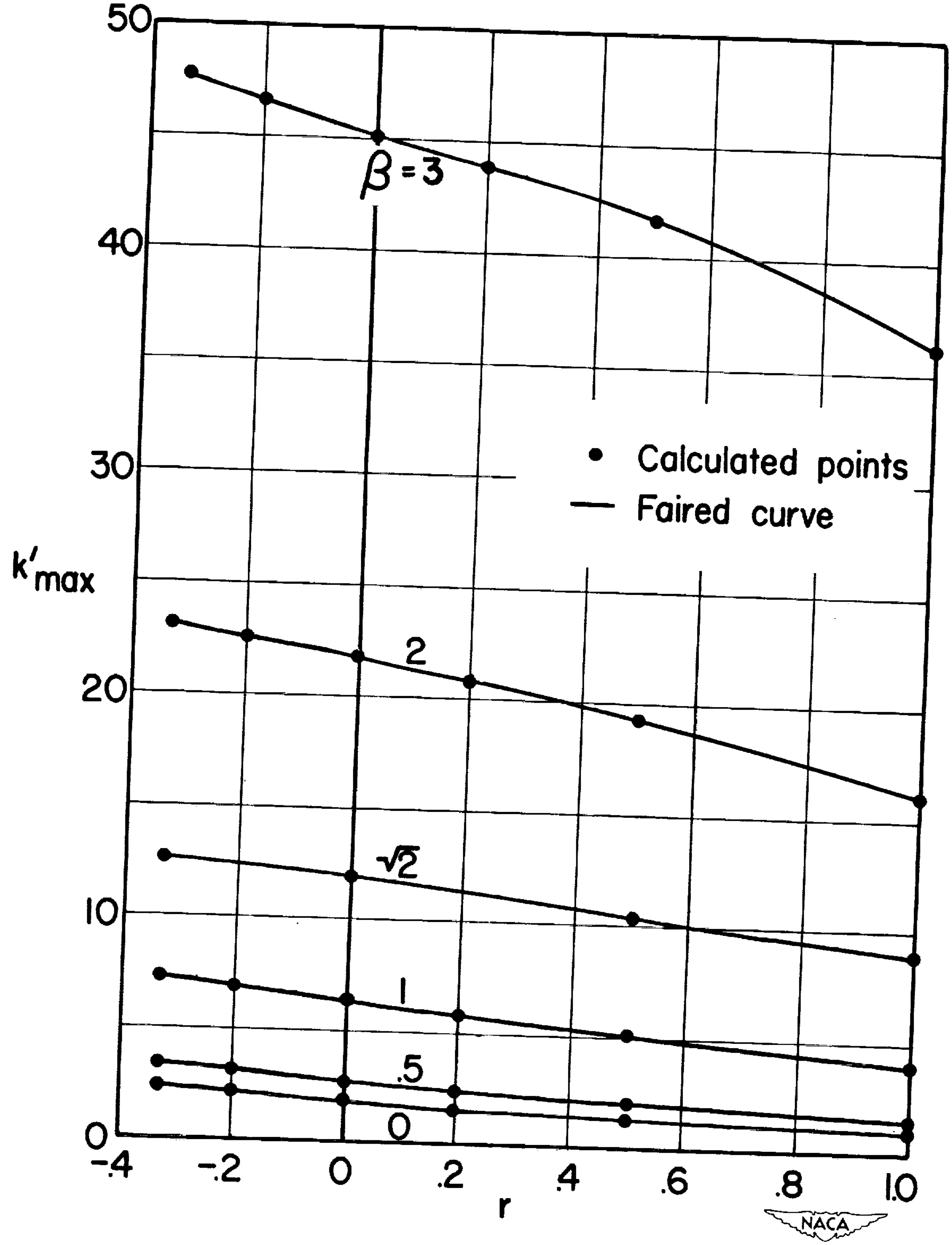

Figure 3.— Coefficient k'_{max} used in calculating buckling stress σ_{max}. $\sigma_{max} = k'_{max}\left(\frac{\pi^2 D}{a^2 t}\right)$

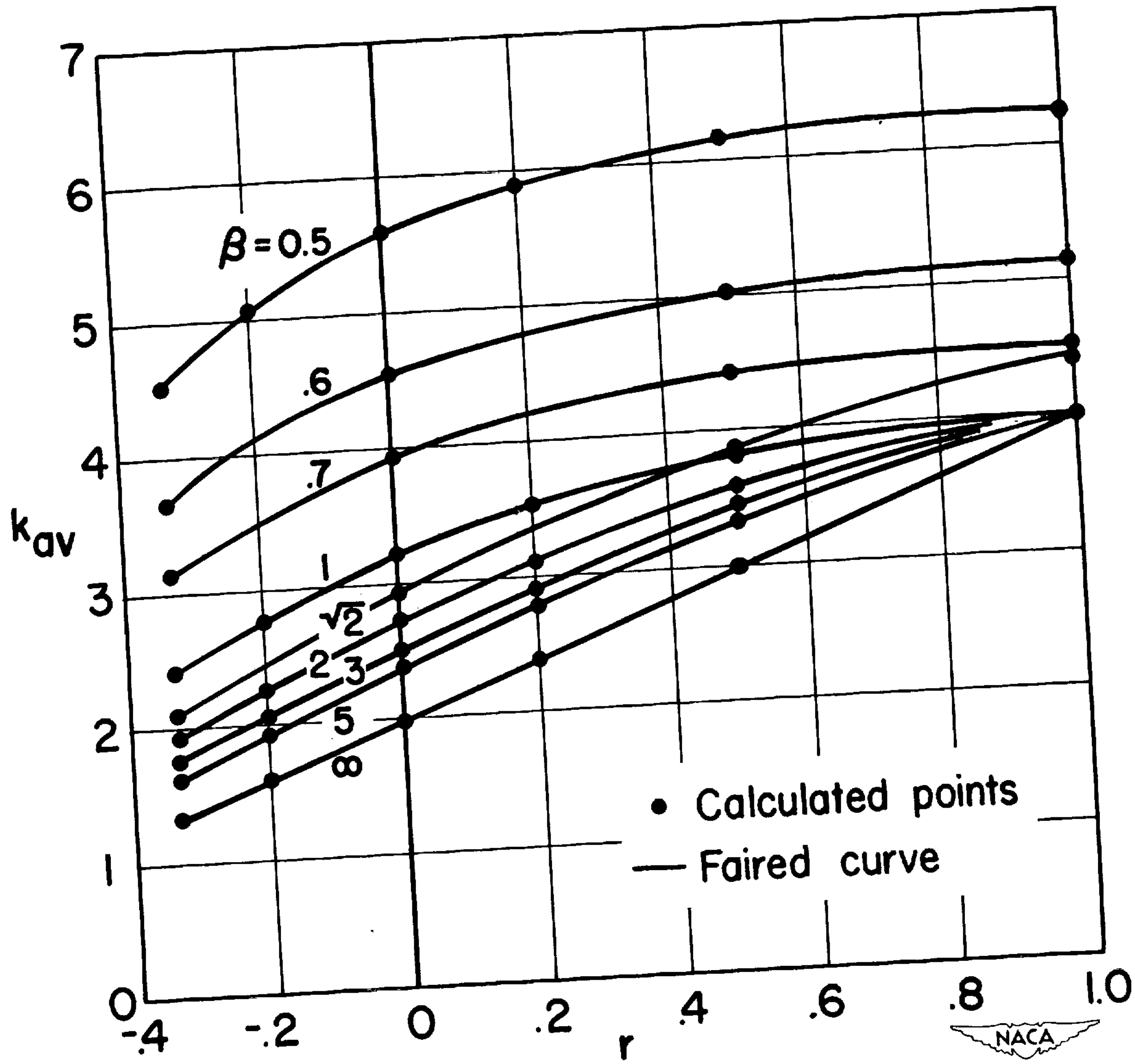

Figure 4.— Coefficient k_{av} used in calculating buckling stress σ_{av}. $\sigma_{av} = k_{av}\left(\frac{\pi^2 D}{b^2 t}\right)$

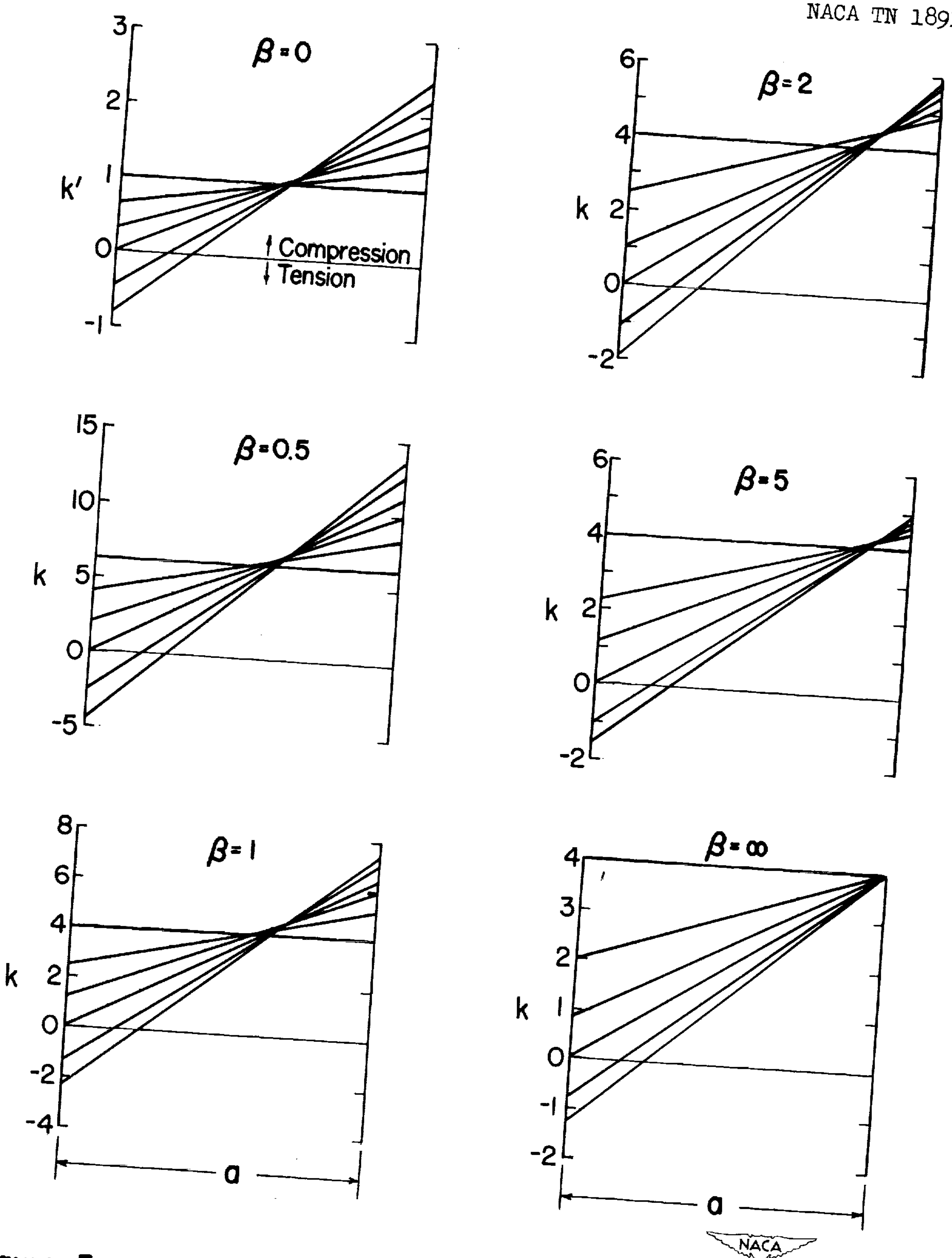

Figure 5.— Longitudinal direct-stress distributions causing buckling.

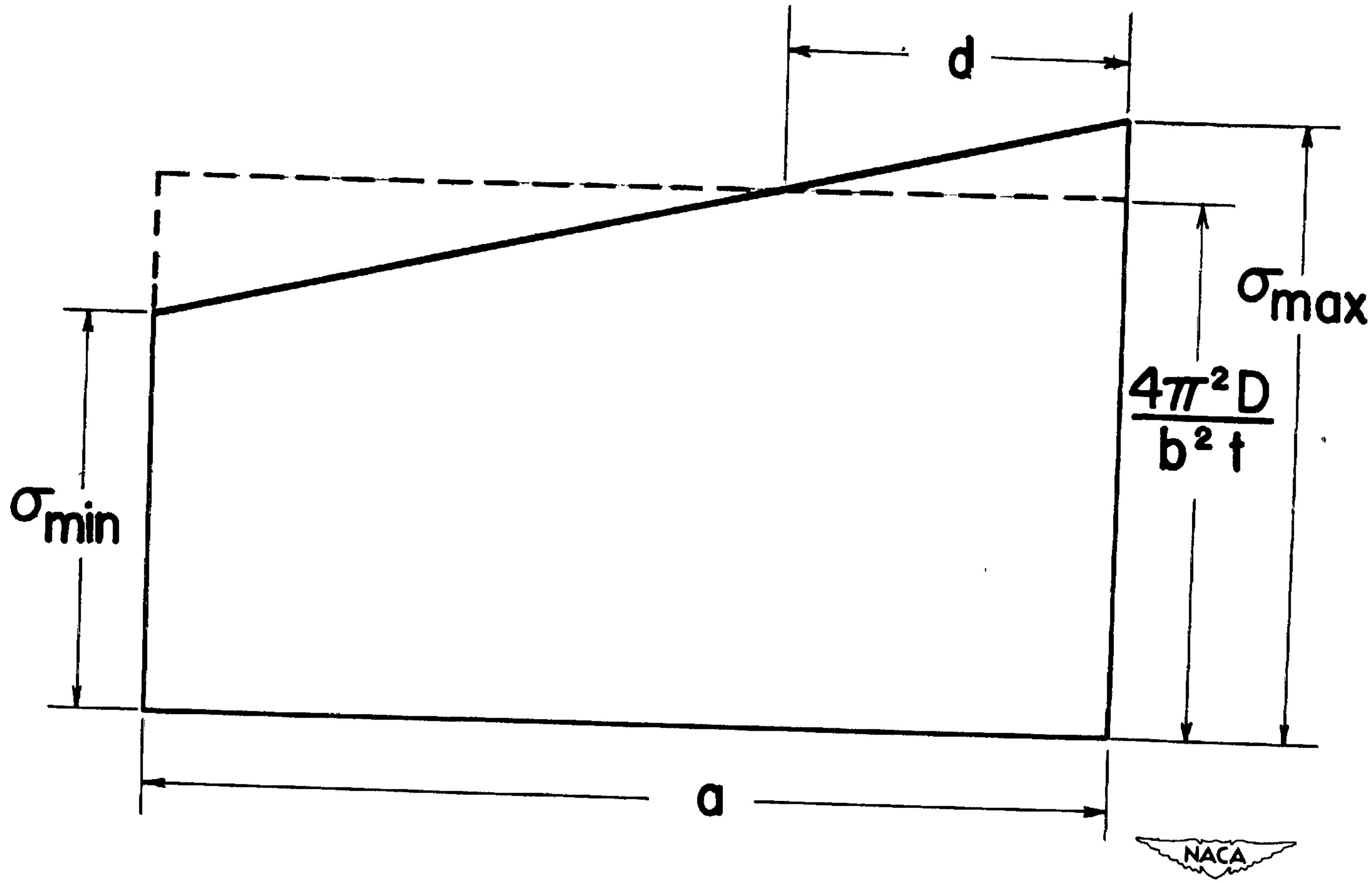

Figure 6.— Longitudinal direct-stress distribution causing buckling, compared with uniform compressive stress required for buckling.